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Considering greenhouse gas emissions in maintenance optimisation

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A R T I C L E I N F O

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A B S T R A C T

Greenhouse gases (GHG) from human activities are the main contributor to climate change since the mid-20th century. Reducing the release of GHG emissions is becoming a thematic research topic in many research disciplines. In the reliability research community, there are research papers relating to reliability and maintenance for systems in power generation farms such as offshore farms. Nevertheless, there is sparse research that aims to optimise maintenance policies for reducing the GHG emissions from systems such as automotive vehicles or building service systems. To fill up this gap, this paper optimises replacement policies for systems that age and degrade and that produce GHG emissions (i.e., exhaust emissions) including the initial manufacturing GHG emissions produced during the manufacturing stage and the emissions generated during the operational stage. Both the exhaust emissions process and the failure process are considered as functions of two time scales (i.e., age and accumulated usage), respectively. Other factors that may affect the two processes such as ambient temperature and road conditions are depicted as random effects. Under these settings, the decision problem is a nonlinear programming problem subject to several constraints. Replacement policies are then developed. Numerical examples are provided to illustrate the proposed methods.

1. Introduction

Many engineering systems, including automotive vehicles and multi-dwelling houses, are faced with a key problem: To balance the trade-off between maintenance price and the greenhouse gas (GHG) emissions. The former ensures the system owners' financial sustainability, and the latter aims to achieve the environmental sustainability required by the society. However, the research on the environmental sustainability in the reliability community has been surprisingly sparse.

In the reliability literature, most of the publications highlight particularly an economical aspect, chiefly through minimising the expected cost, in studies such as redundancy allocation (Reihaneh, Ardakan & Eskandarpour, 2021), system configuration optimization (Yan, Qiu, Peng, & Wu, 2020), maintenance policy optimisation (Gao, Peng, Qu, & Wu, 2020) and warranty policy optimisation (Wang, Li & Xie, 2020; Mitra, 2021), to name a few. A global scale considering an ecological element is becoming more important in many other research communities. The main reason is the global warming is becoming a major issue and problem that requires all research communities to deal with. The nations across the world have reached some agreements such as the Kyoto Protocol and the Paris Agreement. To reduce the GHG emissions is the aim of the binding targets of the Kyoto Protocol and the Paris Agreement is a legally binding international treaty on climate change. This motivates us to incorporate the GHG emissions in the optimisation of maintenance policies.

1.1. Motivating examples

1.1.1. Automotive cars

The GHG emissions during manufacturing a product item are said its 'initial manufacturing emissions'. Once a system is manufactured, its manufacturing emissions are fixed. As such, the longer a product item will be used, the smaller the expected amount of emissions per unit of usage in its lifetime. For example, Berners-Lee and Clark (2010) comment: *If you make a car last to 200,000 miles rather than 100,000, then the emissions for each mile the car does in its lifetime may drop by as much as 50%, because of getting more distance out of the initial manufacturing emissions.*

An engineering system may also produce GHG emissions during its operational stage and the emissions may worsen with its age. Take automotive vehicles as an example, the exhaust emissions of a vehicle depend on a variety of factors including accumulated mileage, its speed (or road type), its age, engine size and weight (Ahlvik, 1997). Among these factors, age and accumu-

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lated mileage are recorded. Kuhns, et al. (2004) suggest that the emissions of particulate matter, carbon monoxide, nitrogen oxide (NO) and hydrocarbons per unit of fuel burned from in-use on-road vehicles increase with vehicle age. More specifically, Chen and Borken-Kleefeld (2016) find that NO_x unit emissions for some diesel automotive vehicles increase with their age and suggest that a deterioration of tailpipe NO_x emissions over 80,000 km of 22% and 10% for two types of technologies, respectively. Zhang, et al. (2017) concludes the trends of the GHG emissions are associated with accumulated mileages.

Noticeably, Sharma and Chung (2015) believe PM emissions increase more drastically in India than in the US with vehicle age. They conclude that *the vehicles in India are often poorly maintained …. We believe that this faster deteriorating also stems due to lack of effective inspection and maintenance systems*. Their comments stress maintenance plays a vital role in deterring the GHG emissions.

1.1.2. Residential buildings

Without any doubt, there is a huge amount of initial manufacturing emissions (or construction emissions) for buildings.

In addition, residential buildings may need to be heated up or cooled down in hot or cold weather conditions. When a building becomes older, hot or cold air may leak from it, and therefore produces exhaust GHG emissions. Estiri and Zagheni (2019) find that an overall increasing profile in energy consumption by age, controlling for income, local climate, and housing age, type, and square footage in the U.S. residential buildings. They also show the growth rate in energy consumption over age is not linear. Aksoezen, Daniel, Hassler, and Kohler (2015) suggest that construction age be a non-linear (i.e., concave that can be modelled by $E_{\text{consumption}} = at^2 + bt$) indicator for energy consumption.

Apparently, GHG missions happen due to energy consumption and maintenance is therefore needed. For instance, the UK government makes advice on ways to save energy, including, *insulating your loft and cavity walls* and *upgrading your boiler* (UK Government 2021).

1.2. Related work and our methods

1.2.1. Maintenance policy optimisation and GHG emissions related work

From the aforementioned discussion, there is a need to develop maintenance policies for such systems. This paper therefore aims to optimise a maintenance policy that considers initial GHG emissions, exhaust emissions and system failures due to other reasons. As such, to incorporate the above three factors in optimisation of maintenance policies, we need to use the nonlinear programming subject to constraints. The exhaust emissions will be modelled by the gamma process and the system failure due to other reasons will be modelled by the nonhomogeneous Poisson process (NHPP).

There is a bulk of research on optimisation of maintenance policies. The reader is referred to De Almeida et al. (2015); Syan and Ramsoobag (2019); de Jonge and Scarf (2020) for literature review, where de Almeida et al. (2015) and Syan and Ramsoobag (2019) review multi-criteria models applied for solving maintenance optimization problems and de Jonge and Scarf (2020) review more than two hundred papers on maintenance modelling and optimization that are published in the period 2001 to 2018.

In the literature, optimisation of maintenance policies considering the GHG emissions has been indirectly investigated. Authors predominately focus on development of maintenance policies for energy production systems such as wind turbines (Li & Coolen 2019, Koukoura, Scheu, & Kolios, 2021) and solar systems (Choe, Guo, Byon, Jin, & Li, 2016, Sayed, EL-Shimy, El-Metwally, & Elshahed, 2020) and for second hand products (Park, Jung, & Park, 2020, Dai, Wei, Wang, He, & He, 2021). It is noted that planning maintenance policies for second hand products can certainly save resources and hence the initial GHG emissions as manufacturing an item inevitably needs energy (Park, Jung, & Park, 2020, Dai, Wei, Wang, He, & He, 2021).

There is some work using different stochastic processes to model the failure process of a system in the literature. For example, Caballe, Castro, Perez, and Lanza-Gutierrez (2015) propose a condition-based maintenance strategy for a system subject to two dependent causes of failure: degradation and sudden shocks. Dong, Liu, Cao, and Bae (2020) optimises maintenance policies for a non-repairable item suffering from both an internal stochastic degradation process and external random shocks, where the deterioration is modelled by the gamma process and the arrival number of external shocks is counted with a NHPP.

1.2.2. Incorporating multiple time scales

For a system such as an automotive car or a residential building, it would be desirable to take into consideration all three time scales: chronological age, operating time, and cumulated usage. The chronological age or chronological time since its last repair can be easily obtained. The operating time of a system, however, may not be available. For example, the accumulated usage or mileage of an automotive vehicle is normally recorded and shown on its odometer. It should be noted that the operating time is not equivalent to the mileage, which is a function of the car speed and operating time.

We regard the accumulated usage as an external time-varying covariate, which is non-decreasing in chronological age t. An external covariate (see Section 1.3.4 in Cook and Lawless (2007)) is one whose values are determined independently of the failure process for that unit, such as an environmental or usage factor. “Non-decreasing” implies that the accumulated usage may stay unchanged during a period when a car is not driven.

In the literature, there are three approaches to analysing the reliability of a system with two time scales: age and accumulated usage (Wu 2012). These approaches are bivariate (Wu, 2014; Yera, Lillo, Nielsen, Ram_rez-Cobo, & Ruggeri, 2021; Gupta & de Chatterjee, 2017), marginal (Zhu, Lu & Zhang, 2021), and composite time scale approaches (Gertsbakh & Kordonsky, 1998). It is worth mentioning that the bivariate approach proposed by Wu (2014) may model the asymmetric phenomenon between the age and usage, where the asymmetric phenomenon means that: if an item’s age is small, its accumulated usage is normally small; however, if an item’s age is large, its accumulated usage may not be large. Meanwhile, regarding the marginal approach, it is interesting to note that Boulter (2009) fitted real datasets and concluded that the relationship between the accumulated vehicle mileage and age is described by a quadratic equation $u = at^2 + bt$ with $a < 0$, where u is the accumulated usage and t is the chronological age. For example, $a = -452.02$ and $b = 15,274$ for petrol cars with engine capacity < 1400cc (Boulter (2009); p. 21). In the research area of warranty management, many authors assume that there is a linear relationship between the age and the accumulated usage, see Dai et al. (2021), for example. This can be true for the early periods of systems as $u = at^2 + bt$ can be approximated by a linear function $u = ct$ ($c > 0$) when t is small.

Regarding the gamma process, Singpurwalla (1995) incorporate environmental stresses in degradation modelling and assume the level of stresses is constant. Lawless and Crowder (2004) further propose that the scale parameter in a gamma process follows the gamma distribution, which can capture the difference between items under study. Other methods on the scale parameter include that the scale parameter is assumed to be a function of covariates, which can be variables such as environmental stresses. The scale parameter can also be regarded as a function of the accelerated stress and the shape parameter is constant (Lawless & Crowder, 2004; Wang, 2009) or a function of environmental covariates (Wang, 2009). Additionally, the gamma prorcess has been used in condition-based maintenance from time to time (see Liu, Pandey, Wang & Zhao, 2021; Andersen, Andersen, Kulahci & Nielsen, 2022; Bautista, Castro & Landesa, 2022, for example) or modelling degradation processes with random effects (Wang, Wang, Hong & Jiang, 2021).

1.2.3. Our methods and novelty

As above discussed, the operational age of an item is not normally recorded. This paper therefore considers the chronological age and accumulated usage, but not the operational age. The marginal and composite time scale approaches will be considered, as elaborated in the following.

We consider a method that estimates the accumulated usage as a function of the chronological age. It then regards the magnitude of the degradation, and the failure intensity function are the functions of linear combinations of the chronological age and accumulated usage, respectively. Since there is a need to use historical data to estimate the relationship between the chronological age and accumulated usage, the drawback of using this method is that the estimated accumulated usage may be biased with a large estimation error. Its strength is that a maintenance policy based on this approach can be determined simply by the chronological time.

We assumes a linear combination of the chronological age and accumulated usage as the moderated time scale, based on which models are developed. The drawback of using this method is its difficulty in management as both the chronological age and accumulated usage must be recorded and monitored, based on which both the chronological age and accumulated usage need to be checked when implementing the maintenance policy. Its strength is that the models may be more accurate as real data are used in modelling and that there is no need to estimate a distribution depicting the joint distribution between the chronological age and accumulated usage.

It is apparent that the composite time scale approach may lead to a parsimonious model as it has fewer parameters than the marginal method.

The paper will use the gamma process to model the degradation process of the exhaust GHG emissions and use the nonhomogeneous Poisson process to model the failure process of the system. It will then incorporate other factors such as the amount of the initial GHG emissions.

Following on the above-discussion, this paper aims to develop a maintenance policy for an item with three factors:

(a) it generates GHG emissions at the operational stage;
(b) it may fail due to other reasons such as ageing or deterioration; and
(c) its initial GHG emissions must be considered in modelling.

To the best of our knowledge, there is little existing research that optimises maintenance policies subject to the constraints of initial GHG emissions and exhaust emissions. Furthermore, as aforementioned, methods of incorporating time varying covariates in the gamma process has rarely been documented in the literature. In addition, the three methods—composite, marginal and bivariate methods—were developed for non-repairable systems, in which only lifetime distributions need to be developed. With these considerations, this paper creates novelty.

1.3. Overview

The remainder of this paper is structured as follows. Section 2 provides assumptions and notations that will be used in the paper. Section 3 develops models for the deterioration process of the GHG emissions and the model of the failure process of the system. Section 4 presents the method to optimise the preventive maintenance policies. Section 5 extends the two time-scale scenarios to multiple cases and discusses relevant maintenance policies. Section 6 provides numerical examples to illustrate the proposed methods. Section 7 concludes the paper and suggests future research.

2. Notations and assumptions

This section makes assumptions and lists notations that will be used in the paper.

2.1. Assumptions

This paper makes the following assumptions.

(A1) The system starts working at time $t = 0$.
(A2) The particulate matter, carbon monoxide, NO and hydrocarbons are referred to as the GHG.
(A3) The system has two failure modes: Modes I and II, both of which deteriorate over two scales: chronological age and accumulated usage. Mode I is the level of its exhaust GHG emissions exceeding a pre-specified threshold, and Mode II is the failure mode due to other reasons. The two failure modes are assumed statistically independent. Both failure processes start from time $t = 0$.
(A4) The degradation progression of Mode I is observable, but it is not repairable. The level of the GHG emissions per unit of time or per unit of usage increases over time and accumulated usage and can be modelled by a gamma process. It is assumed that there are two scenarios. In scenario 1, the level of the GHG emissions is denoted by $X(t,u)$, which is the level of the GHG emissions at chronological age t and accumulated usage u and is the sum of $X_1(t)$ and $X_2(u)$, i.e., $X(t,u) = X_1(t) + X_2(u)$, where $X_1(t)$ is the level of deterioration at chronological age t and $X_2(u)$ is the level of deterioration at accumulated usage u. $X_1(t)$ and $X_2(u)$ are statistically independent. Neither $X_1(t)$ nor $X_2(u)$ can be observed. $X(t,u)$, however, is observable. Both $\{X_1(t) : t > 0\}$ and $\{X_2(u) : u > 0\}$ are gamma processes. In scenario 2, the level of deterioration is observed at a composite time scale $\breve{t} = a_1 t + b_1 u$ and $\{X(\breve{t}) : \breve{t} > 0\}$ is a gamma process.
(A5) The failure progression of Mode II is unobservable. Mode II may experience failures, which can be fixed by repair with its effectiveness assumed to be minimal. It is assumed that there are two scenarios as well. In scenario 1, the failure intensity $\lambda(t,u)$ of Mode II is the sum of $\lambda_1(t)$ and $\lambda_2(u)$, i.e., $\lambda_0(t,u) = \lambda_1(t) + \lambda_2(u)$, where $\lambda_1(t)$ is the failure intensity due to ageing and $\lambda_2(u)$ is the failure intensity due to usage. In scenario 2, the failure intensity $\lambda(\tilde{t})$ of Mode II is an increasing function with respect of a moderated time $\tilde{t}$, which is a linear combination of the chronological age t and accumulated usage, i.e., $\tilde{t} = a_2 t + b_2 u$, for modelling the failure of items. We assume a_2 and b_2 may differ from a_1 and b_1, respectively, as the mechanisms of the degradation processes of modes I and II may be different.
(A6) The system has the initial manufacturing GHG emissions, which is denoted by E_I and is produced during the system's manufacturing stage.
(A7) For the sake of convenience in calculation, the system is inspected every τ_0 units of time or every υ_0 units of usage. That is, the inspection is block-based: for example, an inspection is conducted recently because the time approaches τ_0, another one may be carried out because the accumulated usage approaches ν_0. If the system is down (or failed)

Table 1
Notation table.

i	index of the failure mode; $i = 1$ for failure mode I and $i = 2$ for failure mode II;
T	age of the item under consideration, it is a random variable;
U	accumulated usage of the item, it is a random variable;
$\breve{T}$	composite time scale of failure mode I, it is a random variable;
$\tilde{T}$	composite time scale of failure mode II, it is a random variable;
$t, u, \breve{t}, \tilde{t}$	representing observations of $T, U, \breve{T}$ and $\tilde{T}$, respectively;
$\dot{t}$	a middle variable, which can be t or u;
Z	random effect;
$H(z)$	cumulative distribution function (cdf) of Z;
τ_0	time interval between two consecutive inspections;
ν_0	accumulated usage between two consecutive inspections;
E_I	initial manufacturing emissions;
c_R, c_r, and c_m	cost of replacement (i.e., major PM), minor PM, and minimal repair, respectively

due to Mode II, minimal repair is carried out. If the system is working at the inspection time, the level of GHG emissions of Mode I is checked. Replacement is performed once the optimal replacement time interval is reached, and several other conditions are met. These conditions include the level of the GHG emissions should not be larger than a given threshold, the initial manufacturing GHG emissions per unit of time and per unit of usage should not be larger than given thresholds, respectively.

(A8) The accumulated usage U is non-decreasing over time t.

(A9) The cost of repair is c_m and the cost of a replacement (or, major PM) is c_R, where $c_R > c_m$.

(A10) The time on repair or replacement is negligible.

2.2. Notations

For the sake of convenience, some frequently used notations are listed in Table 1.

3. The process of the GHG emissions and the lifetime distribution of the system

This section aims to derive degradation models for Mode I and the failure process models for Mode II.

3.1. Deterioration of the level of exhaust emissions for Mode I

Let $X_i(\dot{t})$ be the deterioration level at chronological time $\dot{t} = t$ (for $i = 1$) and accumulated usage $\dot{t} = u$ (for $i = 2$), respectively. Assume that $X_i(\dot{t})$ $(i = 1, 2)$ has the following properties:

(a) $X_i(0) = 0$,

(b) the increments $\Delta X_i(\dot{t}) = X_i(\dot{t} + \Delta\dot{t}) - X_i(\dot{t})$ are independent of $\dot{t}$,

(c) $\Delta X_i(\dot{t})$ follows a gamma distribution Gamma$(\alpha_i(\Delta\dot{t}), \beta_i)$ with shape parameter $\alpha_i(\Delta\dot{t})$ and scale parameter β_i, where $\alpha_i(\dot{t})$ is a given monotone increasing function in $\dot{t}$ (for $i = 1$) and u (for $i = 2$), and $\alpha_i(0) = 0$.

$X(\dot{t})$ follows the gamma distribution Gamma$(\alpha_i(\dot{t}), \beta_i)$ with mean $\alpha_i(\dot{t})\beta_i$ and variance $\alpha_i(\dot{t})\beta_i^2$, and its probability density function is given by

$$f\left(x; \alpha_i\left(\dot{t}\right), \beta_i\right) = \frac{\beta_i^{\alpha_i(\dot{t})}}{\Gamma\left(\alpha_i\left(\dot{t}\right)\right)} x^{\alpha_i(\dot{t})-1} e^{-\beta_i x} 1_{\{x>0\}}, \tag{1}$$

where $\Gamma(\cdot)$ is the gamma function: $\Gamma(z) = \int_0^\infty v^{z-1} e^{-v} dv$.

3.1.1. Marginal approach

Since only the chronological age of the system is available, it is more realistic to assume that the system does not operate uninterruptedly over time, instead, it operates in an on and off mode. This is true for vehicle cars: our cars are not always driven on roads. As such, the system alternates between successive on (i.e., driving) intervals and off (i.e., idle intervals), which are denoted by 1 and 0 (suppose the system starts in state 1), where 1 represents that the system is at the on state and 0 at the off state. As such, the usage process of a car can be modelled by an alternating renewal process. This is very similar to boilers in buildings: a boiler may be switched on in cold weather time in a year but it does not work uninterruptedly.

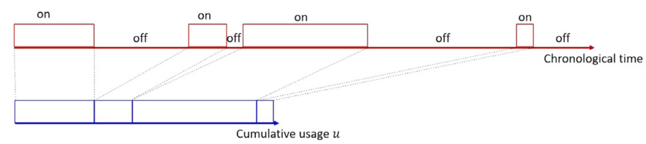

Fig. 1. Relationship between chronological time and accumulated usage.

The degradation process is affected by both the chronological time and the accumulated usage. That is

$$X(t, u) = X_1(t) + X_2(u) = X_1(t) + X_2'(t) \tag{2}$$

Note that $\{X_2(u), u > 0\}$ is a gamma process. But the degradation process $X_2'(t)$ due to usage is not a gamma process in chronological time as the system may at the off state, during which no degradation is caused. As shown in Fig. 1, during the off periods, the system is not operating and does not cause the system to degrade. The top line in Fig. 1 shows the chronological time and the bottom line shows the accumulated usage.

We assume the alternating renewal process composed of two processes: the process of on times $\{U_n : n \geq 1\}$ and the process of off times $\{O_n : n \geq 1\}$ are independent sequences of i.i.d. nonnegative random variables. It is also reasonable to assume that the sequences of pairs $\{(U_n, O_n) : n \geq 1\}$ be i.i.d. non-negative random vectors.

Let Y_{on} and Y_{off} denote the time of the system sojourning at the on state and at the off state, respectively. Denote the CDF of Y_{on} and that of Y_{off} are $W_{\text{on}}(x)$ and $W_{\text{off}}(x)$, respectively. Note that the first bullet in Section 2.1 assumes the system is on at time $t = 0$. For a fixed t, let $\nu_{\text{on}}(t)$ and $\nu_{\text{off}}(t)$ denote the total amount of time the system is on and the total amount of time the system is off during $(0, t)$, respectively. Apparently, $0 \leq \nu_{\text{off}}(t) \leq t$, $0 \leq \nu_{\text{on}}(t) \leq t$, and $\nu_{\text{off}}(t) + \nu_{\text{on}}(t) = t$.

Takacs (1957) showed that the distribution function, $G_{\text{off}}(x|t)$, for the amount of time spent in the off state is

$$G_{\text{off}}(x|t) = P(\nu_{\text{off}}(t) < x) = \sum_{n=0}^{+\infty} W_{\text{off}}^{(n)}(x) \left[W_{\text{on}}^{(n)}(t-x) - W_{\text{on}}^{(n+1)}(t-x)\right], \tag{3}$$

where $W_{\text{off}}^{(n)}(x)$ and $W_{\text{on}}^{(n)}(x)$ denote the n-times iterated convolution of the distribution function $W_{\text{off}}(x)$ and $W_{\text{on}}(x)$, respectively, $W_{\text{on}}^{(0)}(x)=1$ and $W_{\text{off}}^{(0)}(x)=1$ for $x\geq 0$.

Then the distribution function, $G_{\text{on}}(x|t)$, for the amount of time spent in the on state is

$$G_{\text{on}}(x|t)=P(\nu_{\text{on}}(t)<x)=P(t-\nu_{\text{off}}(t)<x)$$
$$=1-\sum_{n=0}^{+\infty}W_{\text{off}}^{(n)}(t-x)\left[W_{\text{on}}^{(n)}(x)-W_{\text{on}}^{(n+1)}(x)\right]. \quad (4)$$

The marginal approach in dealing with modelling on the two time scales in the existing literature normally approximates $g_{\text{on}}(x|t)(=\frac{dG_{\text{on}}(x|t)}{dx})$ by assuming that $g_{\text{on}}(x|t)=p$, where p is a constant with $0\leq p\leq 1$. See Wang and Su (2016) for example. Apparently, $g_{\text{on}}(x|t)$ derived from Eq. (4) provides a more accurate expression than the assumption of $g_{\text{on}}(x|t)=p$. The downside of Eq. (4) is its complexity in expression, which makes it difficult to derive an explicit closed-form expression. Nevertheless, in practice, thanks to the well-established computational mathematics, it is not difficult to develop numerical algorithms when needed.

As can be seen, the expression of $G_{\text{on}}(x)$ is complicating. For example, Barlow and Hunter (1961) give the exact expression of $G_{\text{off}}(x)$ when $W_{\text{on}}(x)$ and $W_{\text{off}}(x)$ are both exponentials: $W_{\text{on}}(x)=1-e^{-\delta_1 x}$ and $W_{\text{off}}(x)=1-e^{-\delta_2 x}$, then, we can derive $G_{\text{on}}(x|t)$:

$$G_{\text{on}}(x|t)=1-e^{-\delta_1 x}\left[1+\sqrt{\delta_1\delta_2 x}\int_0^{t-x}e^{-\delta_2 y}y^{-1/2}I_1\left(2\sqrt{\delta_1\delta_2 xy}\right)dy\right]$$
$$=1-e^{-\delta_1 x}\left[1+\sum\nolimits_{j=0}^{\infty}\frac{(\delta_1 x)^{j+1}}{j!(j+1)!}\gamma(j+1,\delta_2(t-x))\right], \quad (5)$$

where $I_1(x)$ is the Bessel function of order 1 for the imaginary argument defined by $I_1(x)=\sum_{j=0}^{\infty}\frac{(0.5x)^{2j+1}}{j!(j+1)!}$. Then we can obtain the pdf (probability density function) of $G_{\text{on}}(x|t)$ as follows,

$$g_{\text{on}}(x|t)=\frac{dG_{\text{on}}(x|t)}{dx}=e^{-\delta_1 x}\left\{\delta_1-\sum_{j=0}^{\infty}\frac{\delta_1^{j+1}x^j}{j!(j+1)!}\left[(j+1)\gamma(j+1,\delta_2(t-x))-\delta_2 x(\delta_2(t-x))^j e^{-\delta_2(t-x)}\right]\right\}, \quad (6)$$

where $\gamma(s,x)=\int_0^x t^{s-1}e^{-t}dt$ is the lower incomplete gamma function.

In this subsection, we assume that the level of the GHG emissions is observed on the chronological time and follows a gamma process.

Then the distribution function of $X_1(t)$ is given by

$$P(X_1(t)<x)=F_1(x;\alpha_1(t),\beta_1)$$
$$=\int_0^x f(v;\alpha_1(t),\beta_1)dv=\frac{\gamma(\alpha_1(t),x\beta_1)}{\Gamma(\alpha_1(t))}. \quad (7)$$

The distribution function of $X_2(u)$ in the chronological age t' given by

$$P(X_{2'}(t)<x)=F_2(x;\alpha_2(t),\beta_2)$$
$$=\int_0^x\int_0^x f(v;\alpha_2(u),\beta_2)g_{\text{on}}(u|t)dudv. \quad (8)$$

Let T_L be the chronological time at which the level of the GHG emissions exceeds L. That is,

$$T_L=\inf\{t(\geq 0)|X_1(t)+X_{2'}(t)\geq L\}. \quad (9)$$

Note that the right hand side of the above equation is $X_1(t)+X_{2'}(t)\geq L$, instead of $X_1(t)+X_2(u)\geq L$. The value $X_1(t)+X_{2'}(t)\geq L$ has already mapped the accumulated usage onto the chronological time.

Based on the monotonicity of the gamma process, the distribution of T_L can be obtained by

$$F_{T_L}(t)P(T_L<t)=P(X_1(t)+X_{2'}(t)>L)$$
$$=1-\int_0^L F_1(L-x;\alpha_1(t),\beta_1)f_2(x;\alpha_2(t),\beta_2)dx, \quad (10)$$

where is a compound symbol for making definitions, and

$$f_2(x;\alpha_2(u),\beta_2)=\frac{dF_2(x;\alpha_2(t),\beta_2)}{dx}$$
$$=\int_0^t f(x;\alpha_2(u),\beta_2)g_{\text{on}}(u|t)du.$$

3.1.2. Composite time scale approach

Gertsbakh and Kordonsky (1998) propose a method to aggregate both t and u to create a single composite time scale. Although they use the composite scale to model the probability of failure of an item, we borrow this method to model the deterioration level of the exhaust GHG emissions.

In this method, we regard the single composite time scale as a linear combination of the chronological age and the accumulated usage of the system. That is,

$$\breve{T}=a_1T+b_1U. \quad (11)$$

Assume that the level of the GHG emissions is a gamma process in the single composite time scale $\breve{t}(=a_1t+b_1u)$, which is denoted by $\{X(\breve{t}):\breve{t}\geq 0\}$ That is, $X(\breve{t})$ follows the gamma distribution Gamma$(\alpha_3(\breve{t}),\beta_3)$ with mean $\alpha_3(\breve{t})\beta_3$ and variance $\alpha_3(\breve{t})\beta_3^2$, and its probability density function is given by $f(x;\alpha_3(\breve{t}),\beta_3)$.

Suppose that the system needs to be replaced if the level of the GHG emissions exceeds a pre-specified L. Let $\breve{T}_L$ be the moderated time at which the level of the GHG emissions exceeds L. That is $\breve{T}_L=\inf\{t(>0)X|(\breve{t})=L\}$.

Then the distribution function of $X(\breve{t})$ is given by

$$P\left(\breve{T}_L<x\right)=F_3\left(x;\alpha_3\left(\breve{t}\right),\beta_3\right)$$
$$=\int_0^x f\left(v;\alpha_3\left(\breve{t}\right),\beta_3\right)dv=\frac{\gamma\left(\alpha_3\left(\breve{t}\right),x\beta_3\right)}{\Gamma\left(\alpha_3\left(\breve{t}\right)\right)}. \quad (12)$$

The expected value of $X(\breve{t})$ are $E[X(\breve{t})]=\alpha_3(\breve{t})\beta$ and $E[X(\breve{t})]=\alpha_3(\breve{t})\beta^2$, respectively.

Denote

$$F_{\sigma_L}\left(\breve{t}\right)P\left(\breve{T}_L<L\right)=\frac{\gamma\left(\alpha_3\left(\breve{t}\right),L\beta_3\right)}{\Gamma\left(\alpha_3\left(\breve{t}\right)\right)}. \quad (13)$$

The above assumes that $\breve{T}=a_1T+b_1U$, one may also consider other functions such as $\breve{T}=\alpha_0 TU$.

3.2. Failure process of the system due to Mode II

Failure mode II is a failure mode due to other causes of failures. A real system is usually composed of many components. For such a system, its failures can be due to the failures of different components. The failure process of the system can be modelled by a NHPP, as proved by Drenick (1960) and discussed in Wu (2021)

3.2.1. Marginal approach

The marginal approach regards that the scale parameter is a function of the accelerated stress, while the shape parameter stays constant, and the degradation process is affected by both the operational age and the accumulated usage. Unfortunately, the operational age may not be available for analysis. We therefore use the chronological age as a covariate. We assume that the failure intensity is given by

$$\lambda(t,u)=\lambda_1(t)+\lambda_2(u). \tag{14}$$

Then the cdf is given by

$$F(t)=1-\exp\left\{-\int_0^t(\lambda_1(x)+\lambda_2(x)g_{\text{on}}(x|t))dx\right\}. \tag{15}$$

3.2.2. Composite time scale approach

With the composite time scale approach, the moderated life time $\tilde{T}_2$ is expressed by

$$\tilde{T}_2=a_2T+b_2U. \tag{16}$$

Denote $\lambda_3(\tilde{t})$ as the failure intensity function in respect to $\tilde{t}$. Then the cdf is given by

$$F\left(\tilde{t}\right)=1-\exp\left\{-\int_0^{\tilde{t}}\lambda_3(x)dx\right\}, \tag{17}$$

where $\tilde{t}=a_2t+b_2u$.

3.3. Incorporating other external covariates

Ahlvik et al. (1997) suggest that accumulated mileage, ambient temperature, and road conditions are relevant to exhaust emissions. That is, in addition to the accumulated mileage, the level of exhaust emissions of a vehicle also depends on other factors including its speed (or road type), its age, engine size and weight (Ahlvik, 1997). Since it is not easy to record the ambient temperature and road conditions that a vehicle has experienced or will experience, we will add these factors as random effect. These data may not be recorded. As such, we assume a variable Z, which represents the random effect. Denote the probability density function of Z by $h(z)(=\frac{\partial H(z)}{\partial z})$. $H(z)$ is usually assumed a gamma distribution function.

3.3.1. Marginal approach

When the marginal approach is applied, the distribution of T_L can be obtained by

$$\begin{aligned}P(T_L<t)&=P(X_1(t)+X'_2(t)>L)\\&=1-\int_0^L F_1(L-x;\alpha_1(t),\beta_1)f_2(x;\alpha_2(t),z\beta_2)h(z)dxdz.\end{aligned} \tag{18}$$

3.3.2. Composite time scale approach

When the composite time scale approach is adopted, we use the following model to incorporate the random effect:

$$\begin{aligned}P(X_z(\dot{t})>L)=F_{\sigma_L,z}(\dot{t})&=\int_0^\infty\int_L^\infty f\left(x;z\alpha(\dot{t}),\beta\right)dxdH(z)\\&=\int_0^\infty\frac{\gamma\left(z\alpha(\dot{t}),L\beta\right)}{\Gamma\left(z\alpha(\dot{t})\right)}h(z)dz.\end{aligned} \tag{19}$$

4. Maintenance policies

For systems such as automotive cars and boilers in buildings, we will adopt the block replacement policy, with which the system is always at some scheduled time periodically and repaired upon failure between replacements. The block replacement policy is easy to be implemented and widely used in government regulations.

We assume that the repair upon failures between replacements are minimal. That is, such a repair will restore the failure system to the time exactly before the failure occurred.

4.1. Marginal approach

Then for the marginal approach, we aim to find the PM interval to minimise the expected cost, given by

$$E_c(T_0)=\frac{c_R+c_mE[N_0(T_0)]}{T_0}. \tag{20}$$

where $E[N_0(T_0)]=\int_0^{T_0}(\lambda_0(x)+\lambda_1(x)g_{\text{on}}(x|T_0))dx$.

We also need to consider minimising the GHG emissions during the operational stage and the amount of the initial manufacturing emissions. Hence, we set the following optimisation objective and constraints:

$$\begin{aligned}\min_{T_0}\quad & E_c(T_0)\\ s.t.\quad & F_{T_L}(T_0)\ge g_0,\\ & \frac{E_I}{T_0}\le c_t,\end{aligned} \tag{21}$$

where $F_{T_L}(.)$ is from Eq. (10), g_0 may be set to 95%, that is, the probability that the amount of GHG emissions will be smaller than the given limit L is 95%. The second constraint $\frac{E_I}{T_0}\le c_t$ aims to ensure that the initial emissions per unit of time is smaller than a given value.

Lemma 1. Corresponding to the constrained optimization problem one can find that the optimum T_0^* satisfying the following conditions

$$\begin{aligned}&c_R+c_m\int_0^{T_0^*}[\lambda_0(x)+\lambda_1(x)g_{\text{on}}(x|T_0^*)]dx\\&\quad=c_mT_0^*\left(\lambda_0(T_0^*)+\delta_1\lambda_1(T_0^*)e^{-\delta_1T_0^*}\right),\end{aligned} \tag{22}$$

$$F_{T_L}(T_0^*)\ge g_0, \tag{23}$$

$$\frac{E_I}{T_0^*}\le c_t, \tag{24}$$

$$\mu_1(F_{T_L}(T_0^*)-g_0)=0, \tag{25}$$

$$\mu_2\left(\frac{E_I}{T_0^*}-c_t\right)=0, \tag{26}$$

$$\mu_1,\mu_2\ge 0. \tag{27}$$

Proof. According to the Karush–Kuhn–Tucker theorem (Lange, 2013), the optimisation problem shown in Eq. (21) needs to satisfy the condition $\frac{\partial E_c(T_0)}{\partial T_0}|_{T_0=T_0^*}=0$ and conditions from Eq. (23) to (27). Since $g_{\text{on}}(T_0^*|T_0^*)=\delta_1e^{-\delta_1T_0^*}$, $\frac{\partial E_c(T_0)}{\partial T_0}|_{T_0=T_0^*}=0$ can be rewritten as

$$\begin{aligned}&c_R+c_m\int_0^{T_0^*}[\lambda_0(x)+\lambda_1(x)g_{\text{on}}(x|T_0^*)]dx\\&\quad=c_mT_0^*\left(\lambda_0(T_0^*)+\delta_1\lambda_1(T_0^*)e^{-\delta_1T_0^*}\right).\end{aligned} \tag{28}$$

This establishes the lemma.

A government regulation normally requires system owners to preventively maintain their systems on a fixed time period, every 6 months, 12 months, etc, but rarely on a period of a floating point number such as every 1.323 years. Hence, one may consider the following optimisation objective and constraints:

$$\begin{aligned} \min_{n} \quad & E_c(n\tau_0), \\ s.t. \quad & F_{T_L}((n-1)\tau_0) \geq g_0, \\ & \frac{E_I}{n\tau_0} \leq c_t. \end{aligned} \tag{29}$$

The objective is to seek n to minimise the expected function $E_c(n\tau_0)$. The first two constraints are equivalent to $F_{\sigma_L}((n-1)\tau_0) \geq g_0$, which constrains the level of the exhaust emissions within the pre-specified value with a given probability g_0; the third constraint $\frac{E_I}{n\tau_0} \leq c_t$ aims to ensure that the initial emissions per unit of time is smaller than a given value.

4.2. Composite time scale approach

Then for the composite time scale approach, we obtain the expected cost rate due to the failure of Mode II as follows:

$$E_c(\tilde{T}) = \frac{c_R + c_m E[N(\tilde{T})]}{\tilde{T}}, \tag{30}$$

where $E[N(\tilde{T})] = \int_0^{\tilde{t}} \lambda_3(x)dx$.

We can therefore set the following objective and constraints:

$$\begin{aligned} \min_{T_0, U_0} \quad & E_c(a_1T_0 + a_2U_0), \\ s.t. \quad & F_{\sigma_L}(a_1T_0 + a_2U_0) \geq g_0, \\ & \frac{E_I}{T_0} \leq c_t, \\ & \frac{E_I}{U_0} \leq c_u, \end{aligned} \tag{31}$$

where $F_{\sigma_L}(.)$ is from Eq. (13), g_0 may be set to 95%; the second constraint $\frac{E_I}{T_0} \leq c_t$ and the third constraint $\frac{E_I}{U_0} \leq c_u$ aims to ensure that the initial emissions per unit of time and that per unit of usage are smaller than given values, respectively.

Then we have

Lemma 2. The optimum T_0^* and U_0^* satisfy the following conditions:

$$c_R + c_m \Lambda_2(a_1T_0^* + a_2U_0^*) = c_m(a_1T_0^* + a_2U_0^*)\lambda_2(a_1T_0^* + a_2U_0^*), \tag{32}$$

$$F_{\sigma_L}(a_1T_0^* + a_2U_0^*) \geq g_0, \tag{33}$$

$$\frac{E_I}{T_0^*} \leq c_t, \tag{34}$$

$$\frac{E_I}{U_0^*} \leq c_u, \tag{35}$$

$$\mu_1, \mu_2, \mu_3 \geq 0, \tag{36}$$

$$\mu_1(F_{\sigma_L}(a_1T_0^* + a_2U_0^*) - g_0) = 0, \tag{37}$$

$$\mu_2\left(\frac{E_I}{T_0^*} - c_t\right) = 0, \tag{38}$$

$$\mu_3\left(\frac{E_I}{U_0^*} - c_u\right) = 0, \tag{39}$$

Noting that Eq. (32) is obtained from $\frac{E_c(a_1T_0+a_2U_0)}{\partial T_0} = 0$ and $\frac{E_c(a_1T_0+a_2U_0)}{\partial U_0} = 0$, we can establish a similar proof as that in the proof for Lemma 1.

Similar to the optimisation problem discussed in Section 4.1 on the government regulation requirement, we consider discrete maintenance intervals such as 6 months or 12 months. We can then minimise the GHG emissions during the operational stage and the initial manufacturing emission. Hence, we set the following optimisation problem:

$$\begin{aligned} \min_{n,m} \quad & E_c(nw_1\tau_0 + mw_2\upsilon_0), \\ s.t. \quad & F_{\sigma_L}((n-1)w_1\tau_0 + mw_2\upsilon_0) \geq g_0, \\ & F_{\sigma_L}(nw_1\tau_0 + (m-1)w_2\upsilon_0) \geq g_0, \\ & \frac{E_I}{n\tau_0} \leq c_t, \\ & \frac{E_I}{m\upsilon_0} \leq c_u, \end{aligned} \tag{40}$$

where $F_{\sigma_L}(.)$ is from Eq. (13), $\tilde{T} = nw_1\tau_0 + mw_2\upsilon_0$, w_1 and w_2 are weights and $w_1 + w_2 = 1$. g_0 may be set to 95%, that is, the probability that the amount of emissions will be smaller than the given limit L is 95%. The objective is to minimise the expected function $\min_{n,m} E_c(nw_1\tau_0 + mw_2\upsilon_0)$. The first two inequality constraints: $F_{\sigma_L}((n-1)w_1\tau_0 + mw_2\upsilon_0) \geq g_0$ and $F_{\sigma_L}(nw_1\tau_0 + (m-1)w_2\upsilon_0) \geq g_0$), aim to seek optimal m to confine the level of the exhaust emissions with a given probability g_0; the inequality constraint $\frac{E_I}{n\tau_0} \leq c_t$ aims to ensure that the initial emissions per unit of time is smaller than a given value and the third constrain $\frac{E_I}{m\upsilon_0} \leq c_u$ aims to ensure that the initial emissions per unit of usage is less than a given value.

4.3. When Mode I is repairable

Assumption A4) in Section 2.1 assumes that Mode I is not repairable. This section relaxes this assumption and assume that the failures due to Mode I is repairable. To this end, we make the following assumptions.

(A11) Suppose there are two types of preventive maintenance (PM): major and minor, where a major PM is a replacement, after which the item is restored to a good-as-new status. The effectiveness of a minor PM is assumed in the following assumptions, i.e., A12), A13) and A14).

(A12) The minor PM actions are executed at fixed intervals kT_1 with $k = 1, 2, \ldots, N-1$ and the item is replaced at the NT_1. That is, the system undergoes minor PM at successive times $T_1, 2T_1, \ldots, (N-1)T_1$. A major PM, which is a replacement, is conducted at NT_1. Minimal repair is conducted between minor PMs. After either a minor PM or a major PM, the time in the corresponding hazard function or intensity function returns to zero.

(A13) The effectiveness of a minor PM on Mode I is depicted by the geometric process, which is introduced by Lam (1988).
- When the marginal approach is used, the distribution of T_L after the kth minor PM becomes $F_{T_L}(\eta_1^{k-1}t)$ with $\eta_1 > 1$, where $F_{T_L}(t)$ can be obtained by Eq. (10).
- When the composite time scale approach is used, the distribution of $\breve{T}_L$ after the kth minor PM becomes $F_{\sigma_L}(\eta_2^{k-1}\breve{t})$ with $\eta_2 > 1$, where $F_{\sigma_L}(\breve{t})$ can be obtained by Eq. (13).

(A14) The effectiveness of a minor PM on Mode II is depicted by the model proposed by Nakagawa (1988).
- When the marginal approach is used, the failure intensity $\lambda_k(t, u)$ of Mode II after the kth minor PM is the sum of $\lambda_{1,k}(t)$ and $\lambda_{2,k}(u)$, i.e., $\lambda_k(t, u) = \lambda_{1,k}(t) + \lambda_{2,k}(u)$, where $\lambda_{1,k}(t) = \lambda_1(t)\rho_1^k$ is the failure intensity due to

ageing and $\lambda_2(u) = \lambda_2(u)\rho_2^k$ is the failure intensity due to usage, where $\rho_1, \rho_2 \in (1, +\infty)$.

- When the composite time scale is used, the failure intensity $\lambda(\tilde{t})$ of Mode II after the kth minor PM is $\rho_3^k\lambda(\tilde{t})$, where $\rho_3 \in (1, +\infty)$.

(A15) The cost of a minor PM is c_r with $c_R > c_m > c_r$.

4.3.1. Marginal approach

For the marginal approach, we aim to find the minor PM interval to minimise the expected cost, given by

$$E_c(T_1) = \frac{c_R + (N-1)c_r + c_m\int_0^{T_1}\sum_{k=1}^{N}\left(\lambda_1(t)\rho_1^k + \lambda_2(t)\rho_2^k g_{\text{on}}(x|T_1)\right)dx}{NT_1} = \frac{c_R + (N-1)c_r + c_m\int_0^{T_1}\left(\frac{\rho_1-\rho_1^{N+1}}{1-\rho_1}\lambda_1(t) + \frac{\rho_2-\rho_2^{N+1}}{1-\rho_2}\lambda_2(t)g_{\text{on}}(x|T_1)\right)dx}{NT_1} \quad (41)$$

where, in the numerator, the first term c_R is the cost of the major PM at time NT_1, the second term, $(N-1)c_r$, is the sum of the cost of the minor PM at kT_1 with $k = 1, 2, ..., N-1$, and the third term is the sum of the cost on the minimal repair.

Similar to Eq. (21), we set the following optimisation objective and constraints:

$$\begin{aligned} &\min_{T_1} && E_c(T_1) \\ &s.t. && F_{T_L}(T_1) * F_{T_L}(\eta_1 T_1) * \ldots * F_{T_L}\left(\eta_1^{N-1}T_1\right) \geq g_0, \\ & && \frac{E_l}{NT_1} \leq c_t, \end{aligned} \quad (42)$$

where $F_{T_L}(T_1) * F_{T_L}(\eta_1 T_1) * \ldots * F_{T_L}(\eta_1^{N-1}T_1)$, which is the convolution of the cdf's $F_{T_L}(T_1)$, $F_{T_L}(\eta_1 T_1)$, …, and $F_{T_L}(\eta_1^{N-1}T_1)$, is the probability that the sum of the levels of the GHG emissions immediately before the N minor PMs is smaller than L.

4.3.2. Composite time scale approach

For the composite time scale approach, we aim to find the interval of the minor PMs to minimise the expected cost, given by

$$E_c(\tilde{T}) = \frac{c_R + (N-1)c_r + c_m\sum_{k=1}^{N}\int_0^{\tilde{T}_1}\rho_3^k\lambda_3(x)dx}{N\tilde{T}_1} = c_R + (N-1)c_r + \frac{c_m\left(\rho_3 - \rho_3^{N+1}\right)\int_0^{\tilde{T}_1}\lambda_3(x)dx}{1-\rho_3}, \quad (43)$$

The three terms in the numerator in Eq. (43) have similar meanings as those the numerator in Eq. (41).

Similar to Eqs. (31) and (42), we obtain:

$$\begin{aligned} &\min_{T_1, U_1} && E_c(a_1T_1 + a_2U_1), \\ &s.t. && F_{\sigma_L}(a_1T_1 + a_2U_1) * F_{\sigma_L}(\eta_2(a_1T_1 + a_2U_1)) * \\ & && \ldots * F_{\sigma_L}\left(\eta_2^{N-1}(a_1T_1 + a_2U_1)\right) \geq g_0, \\ & && \frac{E_l}{NT_1} \leq c_t, \\ & && \frac{E_l}{NU_1} \leq c_u. \end{aligned} \quad (44)$$

5. Discussion: generalizations

The above content discussed the scenario when there are two scales, time, and accumulated usage, both of which can affect the deterioration process of a failure mode with observable failure progression and the failure of the one with unobservable failure progression. A natural extension is to assume that there are $n+1$ failure modes, in which the first n failure modes are due to n deterioration processes and the last one is the one with unobservable failure progression. Under this setting, we can obtain the following objective function and constraints:

$$\begin{aligned} &\min_{T,U} && E_c(T, U_1, U_2, \ldots, U_n), \\ &s.t. && G(T, U_1, U_2, \ldots, U_n) \geq g_0, \\ & && \frac{E_l}{T} \leq c_t, \\ & && \frac{E_l}{U_i} \leq c_i \ (i = 1, 2, \ldots, n), \end{aligned} \quad (45)$$

where $E_c(T, U_1, U_2, \ldots, U_n)$ is the objective function with $n+1$ intervals (i.e., T, U_i with $i \in \{1, 2, \ldots, n\}$) that needs optimising and $G(T, U_1, U_2, \ldots, U_n)$ is the quantities with interval constraints, and $i \in \{1, 2, \ldots, n\}$.

6. Numerical examples

In this section, we assume $\lambda_0(x) = \xi_{01}\xi_{02}x^{\xi_{02}-1}$ and $\lambda_1(x) = \xi_{11}\xi_{12}x^{\xi_{12}-1}$. Specifically, we denote $\xi_{01} = \xi_{11} = 0.08$, $\xi_{02} = \xi_{12} = 2$, $\delta_1 = 0.65$, $\delta_2 = 0.35$ first in the numerical example and conduct some comparative analysis later. Let $c_R = 100$, $c_m = 50$, $c_p = 20$, and $c_t = c_u = 10$.

6.1. Marginal approach

Under the marginal approach, we have,

$$\begin{aligned} E[N_0(t)] &= \int_0^t (\lambda_1(x) + \lambda_2(x)g_{\text{on}}(x|t))dx \\ &= \xi_{01}t^{\xi_{02}} + \xi_{11}\xi_{12}\delta_1^{1-\xi_{12}}\gamma(\xi_{12}, \delta_1 t) \\ &\quad - \int_0^t \xi_{11}\xi_{12}e^{-\delta_1 x}\sum_{j=0}^{\infty}\frac{\delta_1^{j+1}x^{j+\xi_{12}-1}}{j!(j+1)!}\Big[(j+1)\gamma(j+1, \delta_2(t-x)) \\ &\quad - \delta_2 x(\delta_2(t-x))^j e^{-\delta_2(t-x)}\Big]dx. \end{aligned} \quad (46)$$

Substituting $E[N_0(t)]$ from the above equation into Eq. (20), we can obtain $E_c(T)$. The objective function is $E_c(n\tau_0)$ and the system owner chooses the optimal n^* to minimize the expected cost rate. As for the constraints in Eq. (29), we further assume $L = 20$, $E_l = 20$, $\alpha_1 = \alpha_2 = 2$, $\beta_1 = \beta_2 = 0.5$ and $\tau_0 = 0.5$. As mentioned, the probability that the amount of emissions is greater than the limit L is 95%. In effect, we can refine n from the constraints to several numbers, where $n = 8$ resulting in the lowest expected cost rate, i.e., $E_c(n = 8) = 42.05 < \min(E_c(n = i))$, where i satisfies $F_{T_L}(T_0) \geq g_0$. In other words, we have the optimal $n^* = 8$ and the corresponding objective function as $E_c(n^* = 8) = 42.05$, two decimal places.

We first alter τ from 0.5 to 0.25, 1, and 2 to check the impact of τ on the optimal n^*. The results are shown in Fig. 2. Note that we use the abbreviation *ECR* to denote the expected cost rate. It can be easily shown in Fig. 2a that n^* decreases with τ. This is because with a higher τ and a fixed n and w_1, $F_{T_L}(n\tau_0)$ increases, which makes the probability exceeds the given threshold g_0. Additionally, we will obtain ECR with the same $n\tau_0$, as shown in Fig. 2.

We proceed the analysis by conducting comparative analysis on other parameters. When E_l increases, it is harder for the system owner to find a value n to minimize the objective function due to the g_0 constraint. Nonetheless, if we could accept a higher g_0, n^* will increase, leading to the reduction in the expected cost rate. For instance, if we alter g_0 to 97% while retaining all other parameters the same (e.g., $\tau_0 = 0.5$), the optimal n^* can be obtained by setting $n^* = 9$ and $ECR(n^* = 9) = 39.69$, which is 5.61% lower than the original expected cost rate. In contrast, if we only accept a lower g_0, e.g., 90%, the original solution cannot be employed any

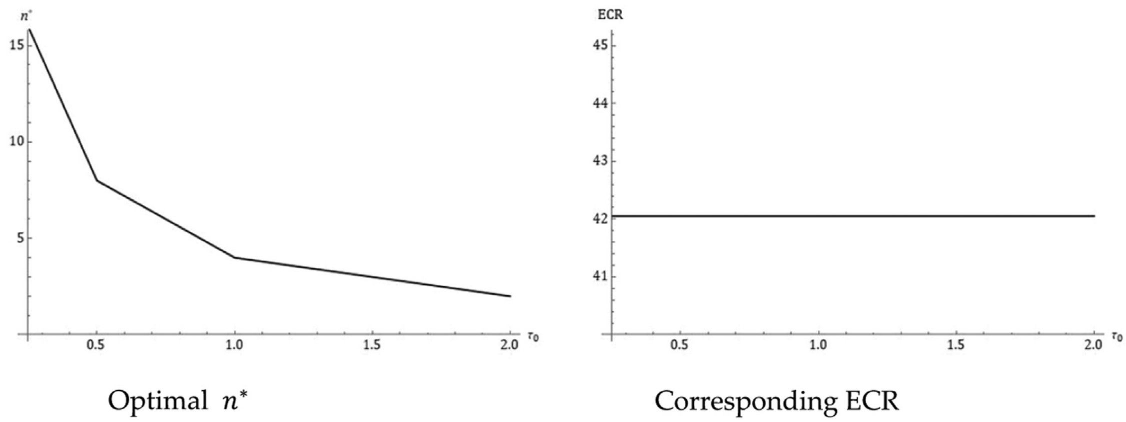

Fig. 2. Optimal n^* and corresponding Expected Cost Rate with respect to τ_0.

more since $F_{T_L}(n^*\tau_0)$ exceeds the altered threshold. The optimal solution thereby goes to $n^*=5$ and the corresponding ECR equals to $ECR(n^*=5)=53.35$, which is 26.87% higher than the original value.

Similarly, if we increase δ_2 and decrease δ_1 while all other parameters remain the same, it will be harder for the system owner to find a value n to meet the constraints. In contrast, if we increase δ_1 or decrease δ_2, i.e., updating $\delta_1=0.5$ and $\delta_2=0.5$, n^* can be chosen to $n^*=7$ and the corresponding expected cost rate can be obtained as $ECR(n^*=7)=46.55$, which is 10.7% higher than the original one.

Besides the previous parameters, the adjustment of shape parameters in the gamma function has impact on the choice of n^* as well. For instance, if we increase α_2, it will be harder for the system owner to find such n^* to satisfy the requirement. Specifically, if we update $\alpha_2=4$, the system owner is not able to find a feasible n. On the contrary, a higher α_1, i.e., $\alpha_1=4$ triggers to $n^*=6$ and $ECR(n^*=6)=48.36$, which is 15.01% higher than the original expected cost rate. Additionally, an augment in L will also make it easy to find n^* while a reduction in L, e.g., $L=10$ makes it only possible to choose $n^*=6$, resulting in $ECR(n^*=6)=48.36$, which is also 15.01% higher than the original case since the alteration of L has no impact on the expected cost rate.

Having checked the parameters that have impacts on the constraints, we now conduct comparative analysis on the objective function. It can be easily derived from the objective function Eq. (29) that c_R and c_m positively impact the expected cost rate, similar to the parameters ξ_{01} and ξ_{02}. As for ξ_{11} and ξ_{12}, we show that both of ξ_{11} and ξ_{12} positively impact it. Specifically, if we increase ξ_{11} to 0.5, n^* remains the same since the constraint is not changed and ECR equals to 47.59, which is 13.15% higher than the original expected cost rate. Similarly, if we increase ξ_{12} to 3, ECR will change to 63.17, which is 50.23% higher than the original one.

6.2. Composite time scale approach

Under the composite time scale approach, we have,

$$E[N(t)]=\int_0^t \lambda_1(x)dx=\xi_{11}t^{\xi_{12}}. \tag{47}$$

Substituting $E[N(t)]$ from the above equation into Eq. (30), we can obtain $E_c(T)$. In effect, the objective function is $E_c(nw_1\tau_0+mw_2\upsilon_0)$ and the system owner chooses the optimal n^* and m^* to minimize the expected cost rate. As for the constraints in Eq. (40), similar to the analysis on marginal approach, we assume $\tau_0=\upsilon_0=1$ and $E_I=20$. Since w_1 and w_2 are weights and $w_1+w_2=1$, we solve the optimization problem with a changing w_1 and show the result in Fig. 3.

The figure is symmetric since we assume $\tau_0=\upsilon_0$. If w_1 is lager (smaller) than 0.5, i.e., the choice of $n(m)$ takes more weight, we have $n^*>m^*(n^*<m^*)$ and vice versa. Additionally, if we increase $\tau_0(\upsilon_0)$ while remaining all other parameters unchanged, the optimal n^* (m^*) decreases and vice versa. We should further note that there are two empty points in Fig. 3. When $w_1=0(w_1=1)$, the choice of $n(m)$ has no impact on the optimal solution as well as the expected cost rate. Additionally, we should note that different combination of (n^*,m^*) may result in the same ECR with the same $nw_1\tau_0+mw_2\upsilon_0$.

Now we take $w_1=0.8$ and $w_2=0.2$ as a benchmark and conduct comparative analysis on other parameters. Note that under the given weights, we have $n^*=5$, $m^*=1$, and $ECR(n^*,m^*)=40.61$. When L increases to 30, the updated optimal solution can be given by $n^*=6$ and $m^*=4$, where the expected cost rate is 40.23 and 0.94% smaller than the original one. In contrast, when L decreases to 10, the updated optimal solution can be given by $n^*=3$ and $m^*=1$, where the expected cost rate is 48.86 and 20.32% higher than the original one. Similarly, the analysis on the objective function shows that c_R, c_m, ξ_{11} and ξ_{12} all positively impact the expected cost rate.

6.3. When Mode I is repairable

We have shown the numerical example when Mode I is not repairable. In this subsection, we proceed our analysis by assuming Mode I is repairable. Following previous assumption, we have $c_R>c_m>c_r=10$ since the cost of minor PM is usually small. Additionally, we assign $\rho_1=1.5$ for tractability while all other parameters remain the same as those in the previous subsection. As for η_1, we consider two scenario where $\eta_1=1.1$ and $\eta_1=1.5$ respectively. Using the same methodology to solve the optimization problem in Eq. (42) with different constraints, we obtain the optimal n^* and corresponding expected cost rate in Fig. 4.

Intuitively, Fig. 4 shares the same pattern as Fig. 2 where the optimal n^* gradually decreases in τ_0. The difference between red line and blue dashed line can be ascribed to the impact of η_1. With a higher η_1 (indicated by the blue dashed line), the constraint on g_0 is more binding, suppressing the possible range for n^*, decreasing the optimal n^*. However, with η_1, ECR increases as there is additional cost that is relevant to Mode I repair. On the contrary, with a lower η_1 (indicated by the red line), a higher n^* can be chosen due to a more flexible constraint. As such, a higher n^* is feasible and the ECR increases as well.

The impacts of other parameters on the optimal n^* and ECR remains the same as that when Mode I is non-repairable, e.g., a lower E_I or g_0 increases the optimal n^* and leads to the decrease in ECR. Furthermore, c_r positively impacts the ECR as a higher cost on minor PM will lead to the increase in the expected cost rate.

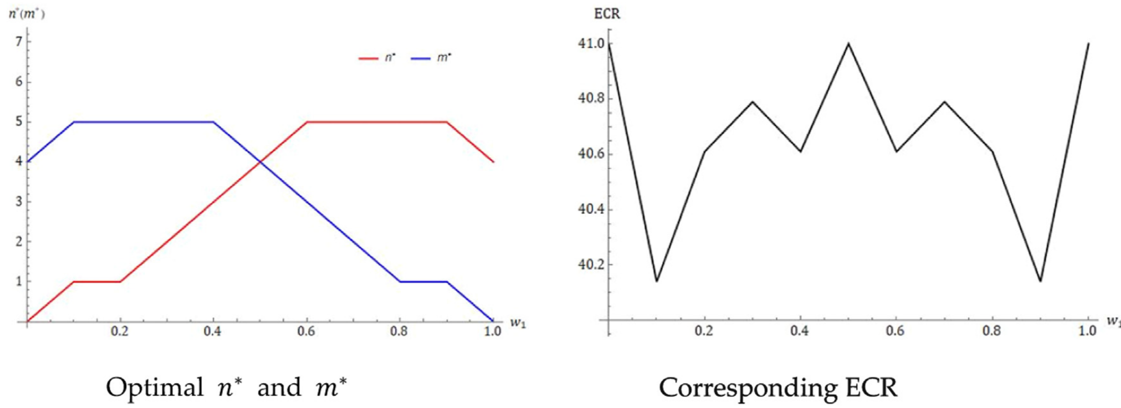

Fig. 3. Optimal n^*, m^* and corresponding Expected Cost Rate with respect to w_1.

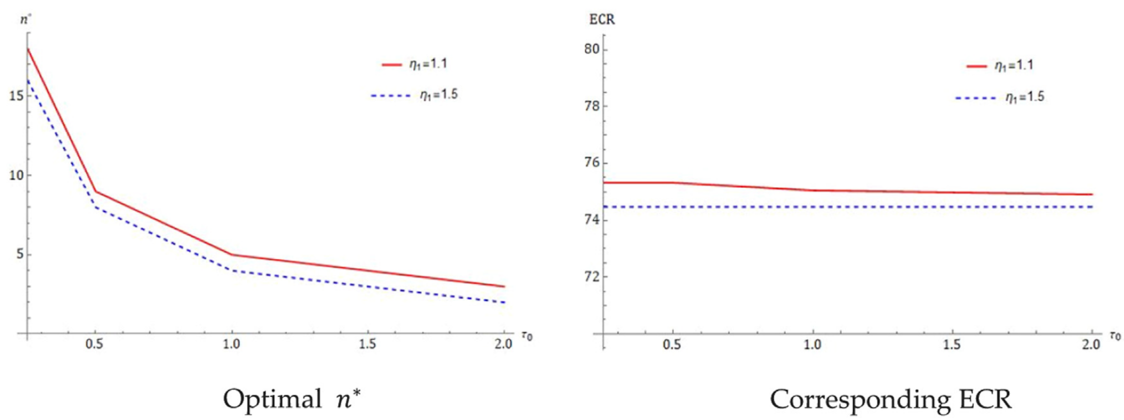

Fig. 4. Optimal n^* and corresponding Expected Cost Rate when Mode I is repairable when Mode I is repairable.

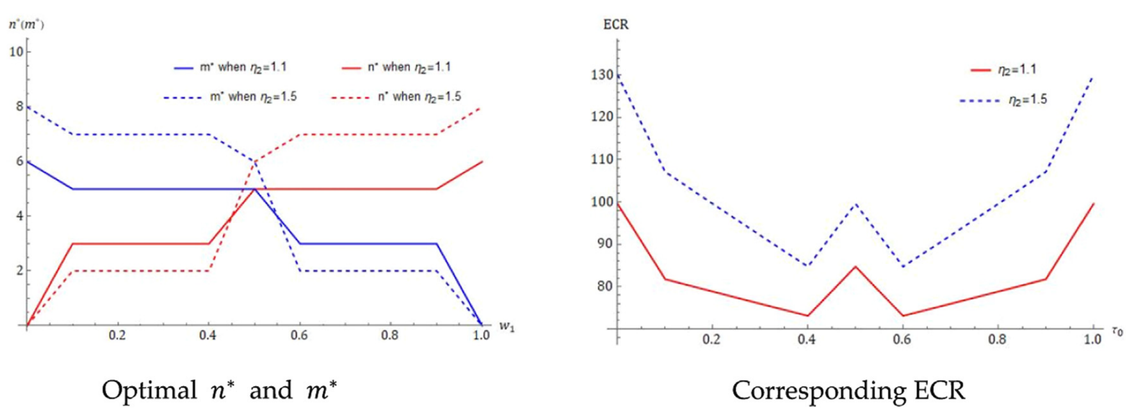

Fig. 5. Optimal n^*, m^* and corresponding Expected Cost Rate when Mode I is repairable.

For instance, if we increase c_r from 10 to 15 when $\eta_1 = 1.1$ and $\tau_0 = 0.5$, ECR increases from 75.07 to 75.47. Differently, if we decrease c_r from 10 to 5, ECR decreases to 73.47.

We continue the analyse by solving the optimization problem in Eq. (44). Similarly, we assign $\rho_2 = \rho_3 = 1.5$ and $c_r = 10$ and perform the results when $\eta_2 = 1.1$ and $\eta_2 = 1.5$ respectively in Fig. 5.

It is not surprising that Fig. 5 shares the same pattern as Fig. 3. We should further note that when $w_1 = 0$ or $w_2 = 0$, the optimal $n^* = 0$ or $m^* = 0$ since the selection of them will have no impact on the ECR. Similarly, when we increase c_r from 10 to 15 (while $w_1 = 0.8, w_2 = 0.2, \eta_2 = 1.1$ and $\tau_0 = \upsilon_0 = 1$), ECR increases from 78.88 to 79.75. And when we decrease c_r from 10 to 5, ECR decreases to 78.01. Similarly, a relative small η_2 indicates a flexible constraint on g_0, making the possible range for n^* and m^* enlarges. Differently, a high η_2 makes the constraint more binding, restricting the selection on n^* and m^*.

Our numerical results illustrate that the proposed model is applicable in reality and is able to provide guidance to the system owner in choosing the optimal PM time period and minimizing the expected cost rate, considering not only the cost related to failure but also cost related to GHG emissions. Specifically, all parameters can be assessed from historical data or other official data released by companies in the same field.

7. Conclusions

This paper incorporated greenhouse gas (GHG) emissions in scheduling maintenance policies. It took the GHG emissions of automotive vehicles as the subject of the research. Aside from relevant costs, three factors were considered: the ageing and deterioration of the system, the increasing exhaust GHG emissions of the system, and the initial manufacturing emissions.

This paper then derived maintenance policies for the situations when the level of the GHG emissions is the total amount of the particulate matter, carbon monoxide, nitrogen oxide and hydrocarbons. Other consideration includes that the minimum amount of those GHG emissions exceeds a pre-specified value. Another extension can be to consider the combination between the chronological time with a non-parametric method such as the kernel method or the spline function method. Alternatively, the non-parametric regression will be on the scale parameter. Time-varying accelerated lifetime models can be applied to the scale parameter of the gamma process. This paper assumed that the two failure modes were statistically independent, which is for the convenience of derivation of the relevant quantities. Our future work will investigate the possibility of relaxing this assumption, use copulas to measure the correlationship, derive relevant quantities such as the first hitting time distributions, and then optimise maintenance policies.

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